DAY SEVENTEEN

Area Bounded by the Curves

Learning & Revision for the Day

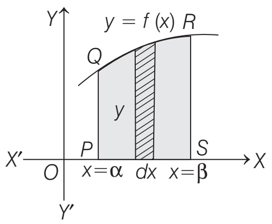
- Curve Area
- Area between a Curve and Lines
- Area between Two Curves

Curve Area

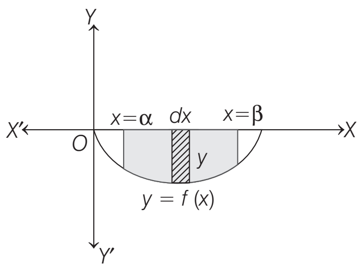
The space occupied by a continuous curve, which is bounded under the certain conditions, is called **curve area** or the area of bounded by the curve.

Area between a Curve and Lines

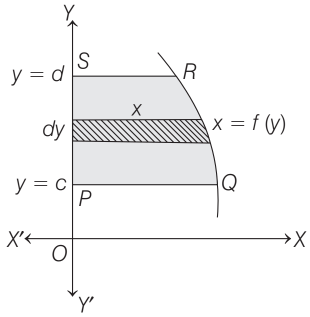
1. The area of region shown in the following figure, bounded by the curve $y = f(x)$ defined on $[\alpha, \beta]$, X-axis and the lines $x = \alpha$ and $x = \beta$ is given by $\int_{\alpha}^{\beta} y\, dx$ or $\int_{\alpha}^{\beta} f(x)\, dx$.



2. If the curve $y = f(x)$ lies below X-axis, then area of region bounded by the curve $y = f(x)$, X-axis and the lines $x = \alpha$ and $x = \beta$ will be negative as shown in the following figure. So, we consider the area as $\left|\int_{\alpha}^{\beta} y\, dx\right|$ or $\left|\int_{\alpha}^{\beta} f(x)\, dx\right|$

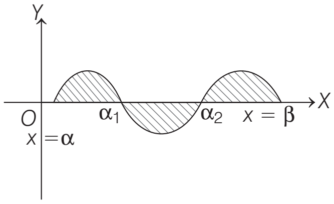


3. Area of region shown in the following figure, bounded by the curve $x = f(y)$, Y-axis and the lines $y = c$ and $y = d$ is given by $\int_c^d f(y)\,dy$ or $\int_c^d x\,dy$



If the position of the curve under consideration is on the left side of Y-axis, then area is given by $\left|\int_c^d f(y)\,dy\right|$.

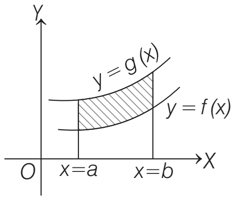
4. If the curve crosses the X-axis number of times, the area of region shown in the following figure, enclosed between the curve $y = f(x)$, X-axis and the lines $x = \alpha$ and $x = \beta$ is given by



$$\int_{\alpha}^{\alpha_1} f(x)\,dx + \left|\int_{\alpha_1}^{\alpha_2} f(x)\,dx\right| + \int_{\alpha_2}^{\beta} f(x)\,dx$$

Area between Two Curves

1. (i) Area of region shown in the following figure, bounded between the curves $y = f(x)$, $y = g(x)$, where $f(x) \le g(x)$, and the lines $x = a$, $x = b$ $(a < b)$ is given by



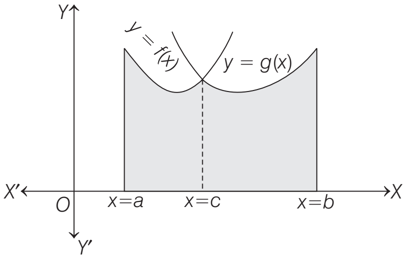
$$\text{Area} = \int_a^b [g(x) - f(x)]\,dx$$

(ii) If $f(x) \ge g(x)$ in $[a, c]$ and $f(x) \le g(x)$ in $[c, b]$ where $a < c < b$, then Area

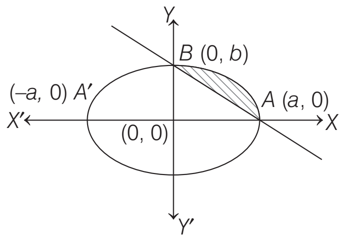
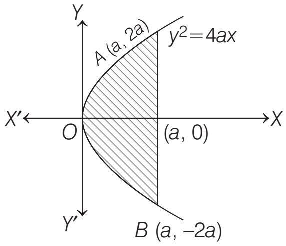
$$= \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx$$

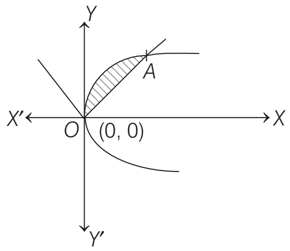
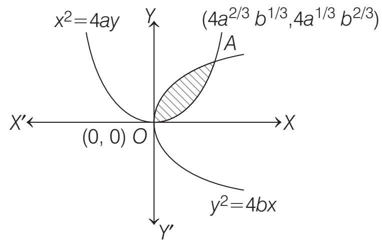
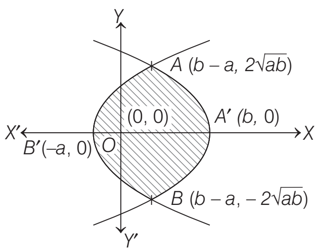
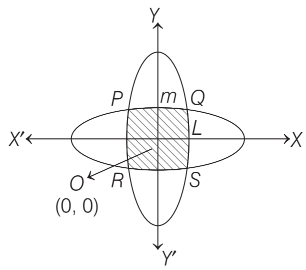
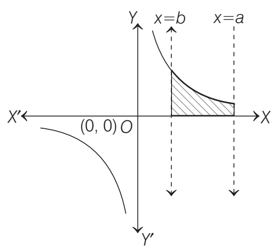
2. Area of region shown in the following figure, bounded by the curves $y = f(x)$, $y = g(x)$, X-axis and lines $x = a$, $x = b$ is given by

$$\text{Area} = \int_a^c f(x)\,dx + \int_c^b g(x)\,dx$$



Area and the shape of some important curves

S.No.	Curves	Point of intersection	Area of shaded region	
(i)	$f(x, y): \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1, \frac{x}{a} + \frac{y}{b} \ge 1$ $\Rightarrow$ or $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \le \frac{x}{a} + \frac{y}{b}$	$A\,(a, 0)$, $B(0, b)$	Area $= ab\,\frac{(\pi - 2)}{4}$ sq units	
(ii)	Parabola $y^2 = 4ax$ and its latusrectum $x = a$	$A\,(a, 2a)$, $B(a, -2a)$	Area $= \frac{8}{3}a^2$ sq units	

(iii)	$f(x, y): y^2 = 4ax$ and $y = \|mx\|$	$O(0, 0), A\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$	Area $= \frac{8a^2}{3m^3}$ sq units	
(iv)	$f(x, y): x^2 = 4ay$ $y^2 = 4bx$	$O(0, 0)$, $A(4a^{2/3}b^{1/3}, 4a^{1/3}b^{2/3})$	Area $= \frac{16}{3}(ab)$ sq units	
(v)	Area bounded by $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$	$A(b - a, 2\sqrt{ab})$ $B(b - a, -2\sqrt{ab})$	Area $= \frac{8}{3}\sqrt{ab}\,(a + b)$ sq units	
(vi)	Common area bounded by the ellipses $\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{1}{a^2b^2}$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2b^2}, 0 < a < b$	$x = y = \frac{1}{\sqrt{a^2 + b^2}}$ $x = y = -\frac{1}{\sqrt{a^2 + b^2}}$	Area = Area of region $PQRS$ $= 4 \times$ Area of $OLQM$ $= \frac{4}{ab}\tan^{-1}\left(\frac{a}{b}\right)$ sq units	
(vii)	If $\alpha, \beta > 0$, $\alpha > \beta$ the area between the hyperbola $xy = p^2$, the X-axis and the ordinates $x = \alpha$, $x = \beta$	—	Area $= p^2 \log\left(\frac{\alpha}{\beta}\right)$	

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 Let $f:[-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^{2} x\, f(x)\, dx$, and R_2 be the area of region bounded by $y = f(x), x = -1$, $x = 2$ and the X-axis. Then

(a) $R_1 = 2R_2$ (b) $R_1 = 3R_2$ (c) $2R_1 = R_2$ (d) $3R_1 = R_2$

2 Let A_1 be the area of the parabola $y^2 = 4ax$ lying between the vertex and the latus rectum and A_2 be the area between the latus rectum and the double ordinate $x = 2a$. Then A_1/A_2 is

(a) $(2\sqrt{2}-1)/7$ (b) $(2\sqrt{2}+1)/7$
(c) $(2\sqrt{2}+1)$ (d) $(2\sqrt{2}-1)$

3 The area of smaller segment cut off from the circle $x^2 + y^2 = 9$ by $x = 1$ is

(a) $\frac{1}{2}(9\sec^{-1}3 - \sqrt{8})$ (b) $9\sec^{-1}(3) - \sqrt{8}$
(c) $\sqrt{8} - 9\sec^{-1}(3)$ (d) None of these

4 The ratio of the area bounded by the curves $y = \cos x$ and $y = \cos 2x$ between $x = 0, \pi/3$ and X-axis, is

(a) $\sqrt{2}:1$ (b) $1:1$ (c) $1:2$ (d) $2:1$

5 The area bounded by the curve $y = f(x)$, X-axis and the lines $x = 1$, $x = b$ is $(\sqrt{b^2+1} - \sqrt{2})$ for all $b > 1$, then $f(x)$ equals to

(a) $\sqrt{x^2+1}$ (b) $\sqrt{x+1}$ (c) $x/\sqrt{x^2+1}$ (d) None of these

6 If the ordinate $x = a$ divides the area bounded by the curve $y = \left(1 + \frac{8}{x^2}\right)$, X-axis and the ordinates $x = 2, 4$ into two equal parts, then a is equal to

(a) 8 (b) $2\sqrt{2}$ (c) 2 (d) $\sqrt{2}$

7 Let the straight line $x = b$ divide the area enclosed by $y = (1-x)^2$, $y = 0$ and $x = 0$ into two parts $R_1 (0 \le x \le b)$ and $R_2 (b \le x \le 1)$ such that $R_1 - R_2 = 1/4$. Then b equals

(a) 3/4 (b) 1/2 (c) 1/3 (d) 1/4

8 The curve $y = a\sqrt{x} + bx$ passes through the point (1, 2) and the area enclosed by the curve, the X-axis and the line $x = 4$ is 8 square units. Then $a - b$ is equal to

(a) 2 (b) – 1 (c) – 2 (d) 4

9 If $y = f(x)$ makes positive intercepts of 2 and 1 unit on x and y-coordinates axes and encloses an area of $\frac{3}{4}$ sq unit with the axes, then $\int_0^2 x f'(x)\, dx$, is

(a) $\frac{3}{2}$ (b) 1 (c) $\frac{5}{4}$ (d) $-\frac{3}{4}$

10 The area of the region (in sq units), in the first quadrant, bounded by the parabola $y = 9x^2$ and the lines $x = 0$, $y = 1$ and $y = 4$, is ➜ **JEE Mains 2013**

(a) $\frac{12}{9}$ (b) $\frac{14}{3}$ (c) $\frac{7}{3}$ (d) $\frac{14}{9}$

11 The area bounded by the curve $y = \ln(x)$ and the lines $y = 0$, $y = \ln(3)$ and $x = 0$ is equal to ➜ **JEE Mains 2013**

(a) 3 (b) 3 ln (3) –2 (c) 3 ln (3)+2 (d) 2

12 The area of the region bounded by the curve $ay^2 = x^3$, the Y-axis and the lines $y = a$ and $y = 2a$, is ➜ **NCERT Exemplar**

(a) $\frac{3}{5}a^2(2\cdot 2^{2/3} - 1)$ sq unit (b) $\frac{2}{5}a\ (2^{2/3} - 1)$ sq unit
(c) $\frac{3}{5}a^2(2^{2/3} + 1)$ sq unit (d) None of these

13 The area of the region bounded by

$y = |x - 3|$ and $y = 2$, is

(a) 4.5 sq units (b) 6.3 sq units
(c) 3.5 sq units (d) None of these

14 The area between the curve $y = 4 - |x|$ and X-axis is

(a) 16 sq units (b) 20 sq units
(c) 12 sq units (d) 18 sq units

15 The area under the curve $y = |\cos x - \sin x|$, $0 \le x \le \frac{\pi}{2}$ and above X-axis is ➜ **JEE Mains 2013**

(a) $2\sqrt{2}$ (b) $2\sqrt{2} - 2$ (c) $2\sqrt{2} + 2$ (d) 0

16 The area of the region enclosed by the curves $y = x$, $x = e$, $y = \frac{1}{x}$ and the positive X-axis is

(a) 1 sq unit (b) $\frac{3}{2}$ sq units (c) $\frac{5}{2}$ sq units (d) $\frac{1}{2}$ sq unit

17 The area bounded by $y = |\sin x|$, X-axis and the lines $|x| = \pi$ is

(a) 2 sq units (b) 3 sq units
(c) 4 sq units (d) None of these

18 For $0 \le x \le \pi$, the area bounded by $y = x$ and $y = x + \sin x$, is

(a) 2 (b) 4
(c) 2π (d) 4π

19 The area (in sq unit) of the region described by $\{(x, y): y^2 \le 2x$ and $y \ge 4x - 1\}$ is ➜ **JEE Mains 2015**

(a) $\frac{7}{32}$ (b) $\frac{5}{34}$
(c) $\frac{15}{64}$ (d) $\frac{9}{32}$

20 The area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$, is

(a) 0 (b) $\frac{32}{3}$ (c) $\frac{16}{3}$ (d) $\frac{8}{3}$

21 If the area bounded by $y = ax^2$ and $x = ay^2$, $a > 0$ is 1, then a is equal to

(a) 1 (b) $\frac{1}{\sqrt{3}}$
(c) $\frac{1}{3}$ (d) None of these

22 The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$ and the straight line $y = 2$, is

(a) $20\sqrt{2}$ (b) $\frac{10\sqrt{2}}{3}$ (c) $\frac{20\sqrt{2}}{3}$ (d) $10\sqrt{2}$

23 The area enclosed the curves $y = x^3$ and $y = \sqrt{x}$ is

(a) $\frac{5}{3}$ sq units (b) $\frac{5}{4}$ sq units
(c) $\frac{5}{12}$ sq unit (d) $\frac{12}{5}$ sq units

24 The area (in sq units) of the region $\{(x, y) : y^2 \geq 2x$ and $x^2 + y^2 \leq 4x, x \geq 0, y \geq 0\}$ is → **JEE Mains 2016**

(a) $\pi - \frac{4}{3}$ (b) $\pi - \frac{8}{3}$
(c) $\pi - \frac{4\sqrt{2}}{3}$ (d) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$

25 The area of the region described by $A = \{(x, y): x^2 + y^2 \leq 1$ and $y^2 \leq 1 - x\}$ is → **JEE Mains 2014**

(a) $\frac{\pi}{2} + \frac{4}{3}$ (b) $\frac{\pi}{2} - \frac{4}{3}$ (c) $\frac{\pi}{2} - \frac{2}{3}$ (d) $\frac{\pi}{2} + \frac{2}{3}$

26 The parabola $y^2 = 2x$ divides the circle $x^2 + y^2 = 8$ in two parts. Then, the ratio of the areas of these parts is

(a) $\frac{3\pi - 2}{10\pi + 2}$ (b) $\frac{3\pi + 2}{9\pi - 2}$
(c) $\frac{6\pi - 3}{11\pi - 5}$ (d) $\frac{2\pi - 9}{9\pi + 2}$

27 The area bounded by the curves $y = 2 - |2 - x|$ and $|x| y = 3$ is

(a) $(5 - 4 \ln 2)/3$ (b) $(2 - \ln 3)/2$
(c) $(4 - 3 \ln 3)/2$ (d) None of these

28 The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$, is

(a) 1 sq unit (b) 2 sq units
(c) $2\sqrt{2}$ sq units (d) 4 sq units

29 The area of the smaller region bounded by the circle $x^2 + y^2 = 1$ and the lines $|y| = x + 1$ is

(a) $(\pi - 2)/4$ (b) $(\pi - 2)/2$
(c) $(\pi + 2)/2$ (d) None of these

30 The area (in sq units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, X-axis and lying in the first quadrant is → **JEE Mains 2013**

(a) 9 (b) 36
(c) 18 (d) $\frac{27}{4}$

31 The area bounded by the curves $y = x^2$ and $y = 2/(1 + x^2)$ is

(a) $(3\pi + 2)/3$ (b) $(3\pi - 2)/3$
(c) $(3\pi - 2)/6$ (d) None of these

32 The area bounded by $y = \tan x$, $y = \cot x$, X-axis in $0 \leq x \leq \frac{\pi}{2}$ is

(a) $3\log 2$ (b) $\log 2$
(c) $2\log 2$ (d) None of these

33 Area of the region bounded by curves $x = 1/2$, $x = 2$, $y = \log_e x$ and $y = 2^x$ is equal to $(4 - \sqrt{2})/\log 2 + b - c \log 2$. Then, $b + c$ equals

(a) 2 (b) – 1
(c) 4 (d) None of these

34 The area bounded by the curve $y = (x + 1)^2$, $y = (x - 1)^2$ and the line $y = \frac{1}{4}$ is

(a) $\frac{1}{6}$ sq unit (b) $\frac{2}{3}$ sq unit
(c) $\frac{1}{4}$ sq unit (d) $\frac{1}{3}$ sq unit

35 The area of bounded region by the curve $y = \log_e x$ and $y = (\log_e x)^2$, is

(a) $3 - e$ (b) $e - 3$
(c) $\frac{1}{2}(3 - e)$ (d) $\frac{1}{2}(e - 3)$

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 Let $f(x)=x^2$, $g(x)=\cos x$ and $h(x)=f(g(x))$. Then area bounded by the curve $y=h(x)$, and X-axis between $x=x_1$ and $x=x_2$, where x_1, x_2 are roots of the equation $18x^2-9\pi x+\pi^2=0$, is

(a) $\pi/12$ (b) $\pi/6$ (c) $\pi/3$ (d) None of these

2 The area bounded by $f(x)=\min(|x|,|x-1|,|x+1|)$ in $[-1,1]$ and X-axis, is (in sq unit)

(a) $\frac{1}{5}$ (b) $\frac{1}{4}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

3 A point $P(x, y)$ moves in such a way that $[|x|]+[|y|]=1$, $[\cdot]=$ G.I.F. Area of the region representing all possible positions of the point P is equal to

(a) 8 (b) 4 (c) 16 (d) None of these

4 The area bounded by the curve $xy^2=4(2-x)$ and Y-axis is

(a) 2π (b) 4π (c) 12π (d) 6π

5 The area of the region $R=\{(x, y): |x|\le|y|$ and $x^2+y^2\le 1\}$ is

(a) $\frac{3\pi}{8}$ sq units (b) $\frac{5\pi}{8}$ sq units (c) $\frac{\pi}{2}$ sq units (d) $\frac{\pi}{8}$ sq unit

6 Area of the region bounded by the parabola $(y-2)^2=x-1$, the tangent to it at the point with the ordinate 3, and the X-axis is given by

(a) 9/2 sq units (b) 9 sq units
(c) 18 sq units (d) None of these

7 Let the circle $x^2+y^2=4$ divide the area bounded by tangent and normal at $(1,\sqrt{3})$ and X-axis in A_1 and A_2. Then $\frac{A_1}{A_2}$ equals to

(a) $\pi/(3\sqrt{3}-\pi)$ (b) $\pi/(3\sqrt{3}+\pi)$
(c) $\pi/(3-\pi\sqrt{3})$ (d) None of these

8 The area (in sq units) of the region $\{(x, y): x\ge 0, x+y\le 3, x^2\le 4y$ and $y\le 1+\sqrt{x}\}$ is

➜ **JEE Mains 2017**

(a) $\frac{7}{3}$ (b) $\frac{5}{2}$ (c) $\frac{59}{12}$ (d) $\frac{3}{2}$

9 The area enclosed between the curves $y=\log_e(x+e)$, $x=\log(1/y)$ and the X-axis is equal to

(a) $2e$ (b) 2 (c) $2/e$ (d) None of these

10 Let $f(x)$ be a real valued function satisfying the relation $f(x/y)=f(x)-f(y)$ and $\lim_{x\to 0}\frac{f(1+x)}{x}=3$. The area bounded by the curve $y=f(x)$, y-axis and the line $y=3$ is equal to

(a) e (b) $2e$ (c) $3e$ (d) None of these

11 The area bounded by the lines $y=2$, $x=1$, $x=a$ and the curve $y=f(x)$, which cuts the last two lines above the first line for all $a\ge 1$, is equal to $\frac{2}{3}[(2a)^{3/2}-3a+3-2\sqrt{2}]$.

Then $f(x)=$

(a) $2\sqrt{2x}, x\ge 1$ (b) $\sqrt{2x}, x\ge 1$
(c) $2\sqrt{x}, x\ge 1$ (d) None of these

12 The area bounded by the curve $y=\cos^{-1}(\sin x)-\sin^{-1}(\cos x)$ and the lines $y=0$, $x=\frac{3\pi}{2}$, $x=2\pi$ is

(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{2}$ (c) π^2 (d) $2\pi^2$

13 Let $f(x)$ be continuous function such that $f(0)=1$, $f(x)-f(x/7)=x/7\ \forall x\in R$. The area bounded by the curve $y=f(x)$ and the coordinate axes is

(a) 2 (b) 3 (c) 6 (d) 9

14 If area bounded by the curves $y=x-bx^2$ and $by=x^2$ is maximum, then b is equal to

(a) 1 (b) -1 (c) $b=\pm 1$ (d) None of these

15 Let A_n be the area bounded by the curve $y=(\tan x)^n$ and the lines $x=0$ and $x=\pi/4$. Then, for $n>2$

(a) $\frac{1}{2n}<A_n<\frac{1}{2n-2}$ (b) $\frac{1}{2n+1}<A_n<\frac{1}{2n-1}$
(c) $\frac{1}{2n+2}<A_n<\frac{1}{2n-2}$ (d) $\frac{1}{2n+2}<A_n<\frac{1}{2n}$

ANSWERS

SESSION 1	**1** (c)	**2** (b)	**3** (b)	**4** (d)	**5** (c)	**6** (b)	**7** (b)	**8** (d)	**9** (d)	**10** (d)
	11 (d)	**12** (a)	**13** (d)	**14** (a)	**15** (b)	**16** (b)	**17** (c)	**18** (a)	**19** (d)	**20** (c)
	21 (b)	**22** (c)	**23** (c)	**24** (b)	**25** (a)	**26** (b)	**27** (c)	**28** (b)	**29** (b)	**30** (a)
	31 (b)	**32** (b)	**33** (c)	**34** (d)	**35** (a)					
SESSION 2	**1** (a)	**2** (d)	**3** (a)	**4** (b)	**5** (c)	**6** (b)	**7** (a)	**8** (b)	**9** (b)	**10** (c)
	11 (a)	**12** (a)	**13** (b)	**14** (c)	**15** (c)					

Hints and Explanations

SESSION 1

1 Given, $R_1 = \int_{-1}^{2} x\, f(x)\, dx$,

and $R_2 = \int_{-1}^{2} f(x)\, dx$ and $f(1-x) = f(x)$

Consider, $R_1 = \int_{-1}^{2} x\, f(x)\, dx$

$= \int_{-1}^{2} (1-x)\, f(1-x)\, dx$

$= \int_{-1}^{2} (1-x)\, f(x)\, dx$

$= R_2 - R_1$

$\therefore\ R_2 = 2R_1$

2 $A_1 = 2\int_0^a 2\sqrt{a}\sqrt{x}\, dx$

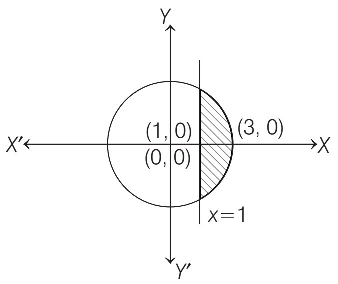
$= 4\sqrt{a} \times \frac{2}{3}[x^{3/2}]_0^a = 8a^2/3$

$A_2 = 2\int_a^{2a} 2\sqrt{a}\sqrt{x}\, dx$

$= 4\sqrt{a} \times \frac{2}{3}[x^{3/2}]_a^{2a} = \frac{8a^2}{3}(2\sqrt{2}-1)$

$\therefore\ A_1/A_2 = \frac{1}{2\sqrt{2}-1} = \frac{(2\sqrt{2}+1)}{7}$

3 Required area, $A = 2\int_1^3 \sqrt{9-x^2}\, dx$



$= 2 \cdot \frac{1}{2}\left[x\sqrt{9-x^2} + 9\sin^{-1}\frac{x}{3}\right]_1^3$

$= \left[9\,\frac{\pi}{2} - \sqrt{8} - 9\sin^{-1}\left(\frac{1}{3}\right)\right]$

$= \left[9\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)\right) - \sqrt{8}\right]$

$= \left[9\cos^{-1}\left(\frac{1}{3}\right) - \sqrt{8}\right]$

$= [9\sec^{-1}(3) - \sqrt{8}]$

4 Here, $A_1 = \int_0^{\pi/3} \cos x\, dx$

$= [\sin x]_0^{\pi/3} = \frac{\sqrt{3}}{2}$

and $A_2 = \int_0^{\pi/3} \cos 2x\, dx$

$= \left[\frac{\sin 2x}{2}\right]_0^{\pi/3} = \frac{\sqrt{3}}{4}$

$\therefore\ A_1 : A_2 = 2 : 1$

5 $\int_1^b f(x)\, dx = \sqrt{b^2+1} - \sqrt{2}$

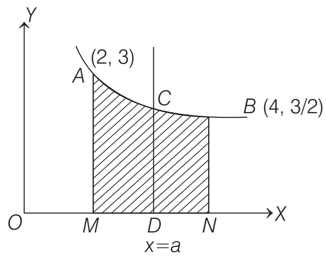
Now, on differentiating both sides w.r.t. b, we get

$\Rightarrow\ f(b) = \frac{2b}{2\sqrt{b^2+1}}\ \forall\, b > 1$

$\Rightarrow\ f(x) = x/\sqrt{x^2+1}$.

6 Area of region $AMNB$

$= \int_2^4 \left(1 + \frac{8}{x^2}\right) dx = \left[x - \frac{8}{x}\right]_2^4 = 4$



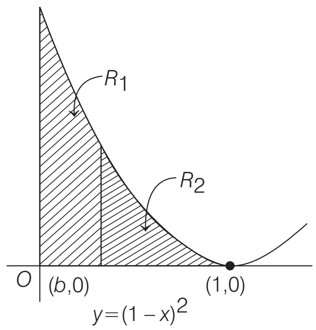
Area of region, $ACDM$

$= \int_2^a \left(1 + \frac{8}{x^2}\right) dx \Rightarrow \left[x - \frac{8}{x}\right]_2^a = 2$

$\Rightarrow\ a = \pm 2\sqrt{2} \Rightarrow a = 2\sqrt{2}$ $[\because a > 0]$

7 We have, $R_1 - R_2 = \int_0^b (1-x)^2\, dx$

$- \int_b^1 (1-x)^2\, dx = \frac{1}{4}$



$\Rightarrow\ \frac{1}{3} - \frac{2(1-b)^3}{3} = \frac{1}{4} \Rightarrow (1-b)^3 = \frac{1}{8}$

$\therefore\ b = \frac{1}{2}$

8 $y = a\sqrt{x} + bx\ (x \ge 0)$.

At $x = 1$, $y = 2$, we get $2 = a + b$ ...(i)

$\int_0^4 (a\sqrt{x} + bx)\, dx = 8$

$\Rightarrow\ \frac{16a}{3} + 8b = 8$ ...(ii)

On solving Eqs. (i) and (ii),

we get $a = 3$ and $b = -1$

$\therefore\ a - b = 4$

9 Clearly, $y = f(x)$ passes through $(2, 0)$ and $(0, 1)$.

$\therefore\ 0 = f(2)$ and $1 = f(0)$

Also, $\int_0^2 f(x)\, dx = \frac{3}{4}$ [given]

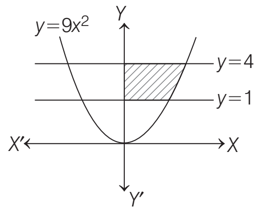
Now, $\int_0^2 x f'(x)\, dx = [xf(x)]_0^2 - \int_0^2 f(x)\, dx$

$\Rightarrow\ \int_0^2 x f'(x)\, dx = [2f(2) - 0f(0)] - \frac{3}{4}$

$\Rightarrow\ \int_0^2 x f'(x)\, dx = 2 \times 0 - 0 \times 1 - \frac{3}{4}$

$= -\frac{3}{4}$

10 Required area $= \int_1^4 \frac{\sqrt{y}}{3}\, dy = \frac{1}{3}\left[\frac{y^{3/2}}{3/2}\right]_1^4$



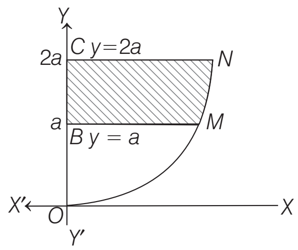
$= \frac{2}{9}(4^{3/2} - 1^{3/2}) = \frac{2}{9} \times 7 = \frac{14}{9}$

11 Required area

$= \int_0^{\log 3} x\,dy = \int_0^{\log 3} e^{y} dy$

$= [e^{y}]_0^{\log 3} = [e^{\log 3} - e^{0}] = 3 - 1 = 2$

12 We have,



Required area =

Area $BMNC = \int_a^{2a} x\,dy$

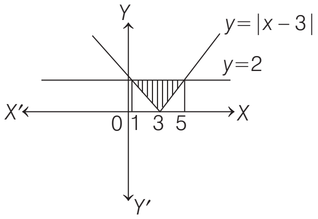
$= \int_a^{2a} a^{1/3} y^{2/3}\, dy = \frac{3a^{\frac{1}{3}}}{5}[y^{5/3}]_a^{2a}$

$= \frac{3a^{\frac{1}{3}}}{5}\left((2a)^{\frac{5}{3}} - a^{\frac{5}{3}}\right)$

$= \frac{3}{5}a^{\frac{1}{3}}a^{\frac{5}{3}}\left((2)^{\frac{5}{3}} - 1\right)$

$= \frac{3}{5}a^2\left(2 \cdot 2^{\frac{2}{3}} - 1\right)$ sq unit

13 The curve $y = |x-3|$ meets the line $y = 2$, when $x = 1$ and $x = 5$

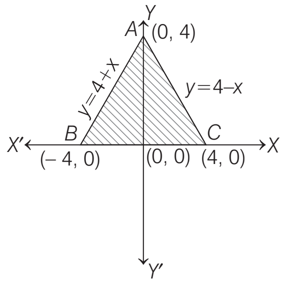


The area of the shaded region

$$= \frac{1}{2} \times 2 \times 4 = 4 \text{ sq units}$$

14 $y = 4 - |x|$ represents two curves as,

$$y = \begin{cases} 4 + x, & x < 0 \\ 4 - x, & x > 0 \end{cases}$$



The area of the shaded portion

$$= \frac{1}{2} \times 8 \times 4 = 16 \text{ sq units}$$

15 Required area

$$= \int_0^{\pi/2} |\cos x - \sin x| dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx$$

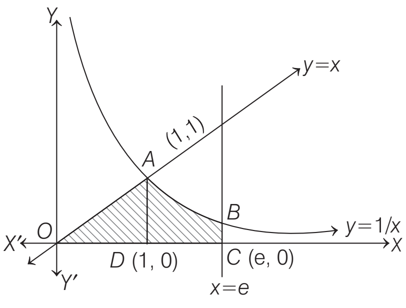
$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= (\sqrt{2} - 1) - (1 - \sqrt{2}) = 2\sqrt{2} - 2$$

16 Given, $y = x$, $x = e$ and $y = \frac{1}{x}$, $x \ge 0$

Since, $y = x$ and $x \ge 0 \Rightarrow y \ge 0$

$\therefore$ Area to be calculated in 1st quadrant shown in figure



$\therefore$ Required area = Area of ΔODA + Area of $DABCD$

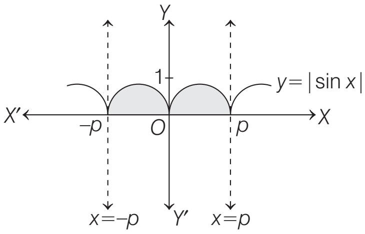
$$= \frac{1}{2}(1 \times 1) + \int_1^e \frac{1}{x} dx = \frac{1}{2} + [\log|x|]_1^e$$

$$= \frac{1}{2} + [\log|x|]_1^e = \frac{1}{2} + 1 = \frac{3}{2} \text{ sq units}$$

17 We have,

$$y = |\sin x| = \begin{cases} \sin x, & \text{if } x \ge 0 \\ -\sin x, & \text{if } x < 0 \end{cases}$$

and $|x| = \pi \Rightarrow x = \pm\pi$



Now, required area $= 2\int_2^{\pi} \sin x dx$

$= 2[-\cos x]_0^{\pi}$

$= -2[\cos \pi - \cos 0]$

$= -2[-1 - 1] = 4$ sq units

18 Given, curves $y = x$ and $y = x + \sin x$, which intersect at $(0, 0)$ and (π, π).

$$\therefore \text{ Area, } A = \int_0^{\pi} (x + \sin x) dx - \int_0^{\pi} x dx$$

$$= \int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi}$$

$$= -\cos \pi + \cos 0$$

$$= -(-1) + 1 = 2$$

19 Given region is

$$\{(x, y) : y^2 \le 2x \text{ and } y \ge 4x - 1\}$$

$y^2 \le 2x$ represents a region inside the parabola $y^2 = 2x$ ...(i)

and $y \ge 4x - 1$ represents a region to the left of the line $y = 4x - 1$ ...(ii)

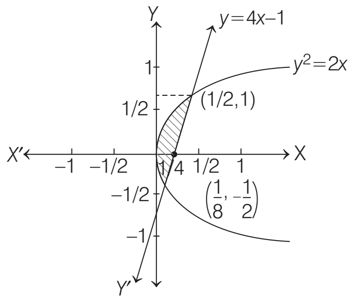
The point of intersection of the curves (i) and (ii) is given by

$(4x - 1)^2 = 2x \Rightarrow 16x^2 + 1 - 8x = 2x$

$\Rightarrow 16x^2 - 10x + 1 = 0$

$\Rightarrow x = \frac{1}{2}, \frac{1}{8}$

So, the points where these curves intersect are $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{1}{8}, -\frac{1}{2}\right)$



Hence, required area

$$= \int_{-\frac{1}{2}}^{1} \left(\frac{y+1}{4} - \frac{y^2}{2}\right) dy$$

$$= \frac{1}{4}\left(\frac{y^2}{2} + y\right)_{-1/2}^{1} - \frac{1}{6}(y^3)_{-1/2}^{1}$$

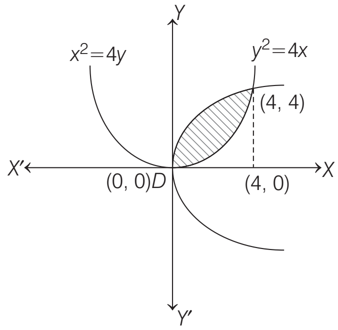
$$= \frac{1}{4}\left\{\left(\frac{1}{2} + 1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right\} - \frac{1}{6}\left\{1 + \frac{1}{8}\right\}$$

$$= \frac{1}{4}\left\{\frac{3}{2} + \frac{3}{8}\right\} - \frac{1}{6}\left\{\frac{9}{8}\right\}$$

$$= \frac{1}{4} \times \frac{15}{8} - \frac{3}{16} = \frac{9}{32}$$

20 For the point of intersection of

$$y^2 = 4x \text{ and } x^2 = 4y$$



Substitute $y = \frac{x^2}{4}$ in $y^2 = 4x$

$$\Rightarrow \left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow x^4 = 4^3 x \Rightarrow x = 0, 4$$

$\therefore$ Area bounded between curves

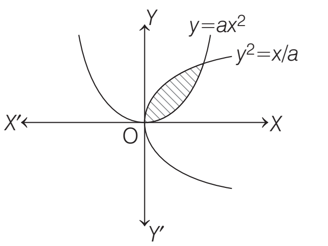
$$= \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right) dx$$

$$= \left[2 \cdot \frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$$

$$= \frac{4}{3} \cdot (4)^{3/2} - \frac{(4)^3}{12}$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$$

21 The intersection point of two curves is $\left(\frac{1}{a}, \frac{1}{a}\right)$.

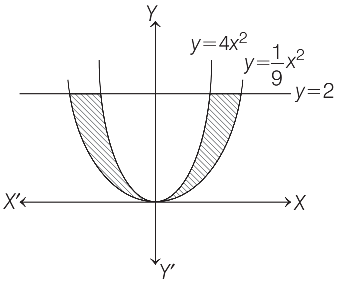


$$\therefore \text{ Area, } A = \int_0^{1/a} \left(\sqrt{\frac{x}{a}} - ax^2\right) dx$$

⇒ $1 = \frac{1}{\sqrt{a}}\cdot\frac{2}{3}[x^{3/2}]_0^{1/a} - \frac{a}{3}[x^3]_0^{1/a}$

⇒ $a^2 = \frac{1}{3} \Rightarrow a = \frac{1}{\sqrt{3}}$

22

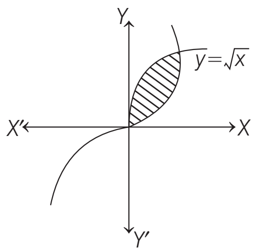


Required area $= 2\int_0^2 \left(3\sqrt{y} - \frac{\sqrt{y}}{2}\right)dy$

$= 2\int_0^2 \left(\frac{5}{2}\sqrt{y}\right)dy = 5\left[\frac{y^{3/2}}{\frac{3}{2}}\right]_{y=0}^{y=2}$

$= \frac{10}{3}(2^{3/2} - 0) = \frac{20\sqrt{2}}{3}$

23 Clearly, the intersection of two curves $y = x^3$ and $y = \sqrt{x}$ are given by $x = 0$ and $x = 1$.



$\therefore A = \left|\int_0^1 (x^3 - \sqrt{x})\,dx\right| = \left|\left[\frac{x^4}{4} - \frac{2x^{3/2}}{3}\right]_0^1\right|$

$= \left|\left[\frac{1}{4} - \frac{2}{3}\right]\right| = \frac{5}{12}$ sq unit

24 We have, $x^2 + y^2 \le 4x$ and $y^2 \ge 2x$

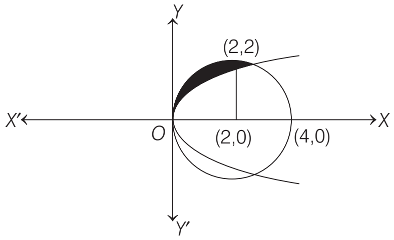
To find point of intersection, substitute $y^2 = 2x$ in $x^2 + y^2 = 4x$

$x^2 + y^2 = 4x \Rightarrow x^2 + 2x = 4x$

⇒ $x^2 - 2x = 0 \Rightarrow x(x-2) = 0$

⇒ $x = 0$ or $x = 2$

⇒ $y = 0$ or $y = 2$



Required area $= \int_0^2 (y_1 - y_2)\,dx$

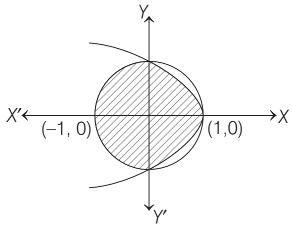
$= \int_0^2 Y_{\text{circle}} - \int_0^2 Y_{\text{parabola}}\,dx$

$= \frac{\pi r^2}{4} - \int_0^2 \sqrt{2}(x)^{1/2}\,dx$

$= \frac{\pi \times 4}{4} - \sqrt{2}\,\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^2$

$= \pi - \frac{2\sqrt{2}}{3}(2^{3/2} - 0) = \pi - \frac{8}{3}$

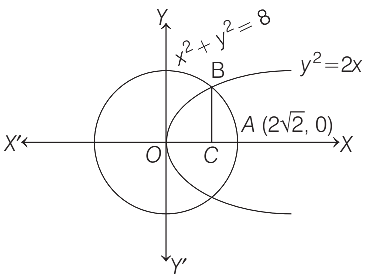
25 Given,
$A = \{(x, y): x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$



Required area $= \frac{1}{2}\pi r^2 + 2\int_0^1 (1 - y^2)\,dy$

$= \frac{1}{2}\pi(1)^2 + 2\left(y - \frac{y^3}{3}\right)_0^1 = \frac{\pi}{2} + \frac{4}{3}$

26



Let the area of the smaller part be A_1 and that of the bigger part be A_2. We have to find $\frac{A_1}{A_2}$.

The point B is a point of intersection (lying in the first quadrant) of the given parabola and the circle, whose coordinates can be obtained by solving the two equations $y^2 = 2x$ and $x^2 + y^2 = 8$.

⇒ $x^2 + 2x = 8 \Rightarrow (x-2)(x+4) = 0$

⇒ $x = 2, -4$

$x = -4$ is not possible as both the points of intersection have the same positive x-coordinate. Thus, $C \equiv (2, 0)$.

Now, $A_1 = 2$ [Area $(OBCO)$ + Area $(CBAC)$]

$= 2\left[\int_0^2 y_1\,dx + \int_2^{2\sqrt{2}} y_2\,dx\right]$,

where, y_1 and y_2 are respectively the values of y from the equations of the parabola and that of the circle.

⇒ $A_1 = 2\left[\int_0^2 \sqrt{2x}\,dx + \int_2^{2\sqrt{2}} \sqrt{8 - x^2}\,dx\right]$

⇒ $A_1 = 2\left[\sqrt{2}\cdot\frac{2}{3}\cdot x^{3/2}\right]_0^2$

$+ 2\left[\frac{x}{2}\sqrt{8 - x^2} + \frac{8}{2}\sin^{-1}\frac{x}{2\sqrt{2}}\right]_2^{2\sqrt{2}}$

$= \frac{16}{3} + 2\left[2\pi - \left(2 + 4 \times \frac{\pi}{4}\right)\right]$

$= \left(\frac{4}{3} + 2\pi\right)$ sq units

Clearly, area of the circle $= \pi(2\sqrt{2})^2$

$= 8\pi$ sq units

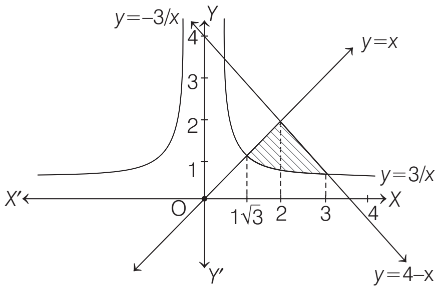
Now, $A_2 = 8\pi - A_1 = 6\pi - \frac{4}{3}$

and the required ratio, $\frac{A_1}{A_2}$ is

$= \frac{\frac{4}{3} + 2\pi}{6\pi - \frac{4}{3}} = \frac{2 + 3\pi}{9\pi - 2}$

27 $y = 2 - |2 - x| = \begin{cases} x & : x < 2 \\ 4 - x & : x \ge 2 \end{cases}$

and $y = \frac{3}{|x|} \Rightarrow y = \begin{cases} -3/x & : x < 0 \\ 3/x & : x > 0 \end{cases}$



⇒ $x = 3/x \Rightarrow x = \sqrt{3}$

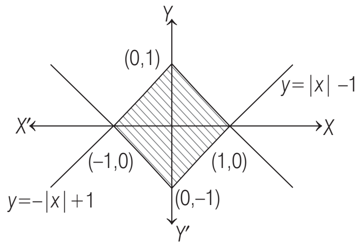
and $4 - x = 3/x \Rightarrow x = 3\ (x > 2)$.

∴ Required area

$= \int_{\sqrt{3}}^2 \left(x - \frac{3}{x}\right)dx + \int_2^3 \left(4 - x - \frac{3}{x}\right)dx$

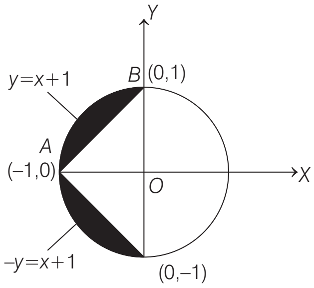
$= (4 - 3\log 3)/2$

28 The region is clearly square with vertices at the points $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$.



So, its area $= \sqrt{2} \times \sqrt{2} = 2$ sq units

29 Due to symmetry, required area



$$= 2\int_{-1}^{0} [\sqrt{1 - x^2} - (x + 1)]\, dx$$

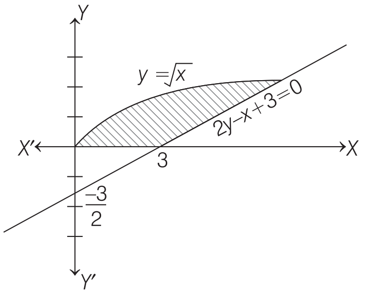
or Area = 2 [area of sector *AOBA* – Δ*AOB*]

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{(\pi - 2)}{2}$$

30 Given curves are

$$y = \sqrt{x} \qquad \text{...(i)}$$

and $2y - x + 3 = 0$ ...(ii)



On solving Eqs. (i) and (ii) , we get

$$2\sqrt{x} - (\sqrt{x})^2 + 3 = 0$$

$$\Rightarrow \quad (\sqrt{x})^2 - 2\sqrt{x} - 3 = 0$$

$$\Rightarrow \quad (\sqrt{x} - 3)(\sqrt{x} + 1) = 0$$

$$\Rightarrow \quad \sqrt{x} = 3,$$

[$\because \sqrt{x} = -1$ is not possible]

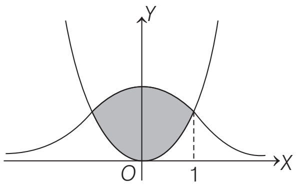
$\therefore\ y = 3$

$\therefore$ Required area $= \int_0^3 (x_2 - x_1)\, dy$

$$= \int_0^3 \{(2y + 3) - y^2\}\, dy$$

$$= \left[y^2 + 3y - \frac{y^3}{3}\right]_0^3 = 9 + 9 - 9 = 9$$

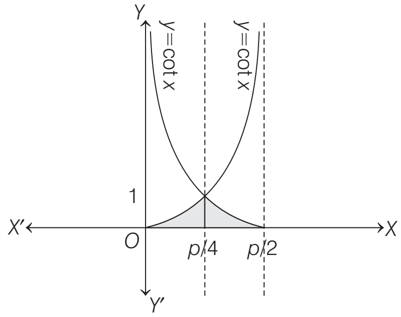
31 Curves $x^2 = y$ and $y = 2/(1 + x^2)$ are symmetrical about *Y*-axis.



$$\text{Area} = 2\int_0^1 \left(\frac{2}{1 + x^2} - x^2\right) dx$$

$$= \left[4\tan^{-1} x - \frac{2x^3}{3}\right]_0^1 = \pi - 2/3 = \frac{3\pi - 2}{3}$$

32 Clearly, the two curves will intersect at $\left(\frac{\pi}{4}, 1\right)$.



Now, required area

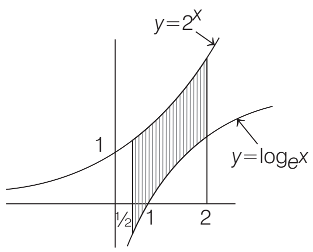
$$= \int_0^{\pi/4} \tan x\, dx + \int_{\pi/4}^{\pi/2} \cot x\, dx$$

$$= [\log \sec x]_0^{\pi/4} + [\log \sin x]_{\pi/4}^{\pi/2}$$

$$= [\log \sqrt{2} - 0] + [0 + \log \sqrt{2}]$$

$$= 2\log \sqrt{2} = \log 2 \text{ sq units}$$

33 Required area



$$= \int_{1/2}^{2} (2^x - \log_e x)\, dx$$

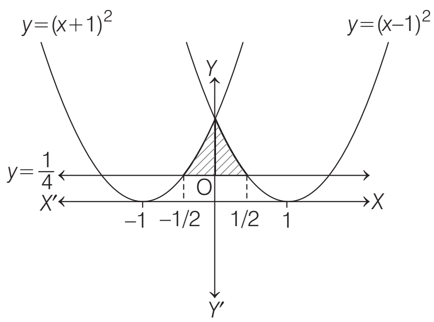
$$= \left[\frac{2^x}{\log 2} - x\log x + x\right]_{1/2}^{2}$$

$$= \frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}$$

$$= \frac{4 - \sqrt{2}}{\log 2} + b - c\log 2$$

$$\Rightarrow \quad b = 3/2, c = 5/2 \Rightarrow b + c = 4$$

34



$\therefore$ Required area

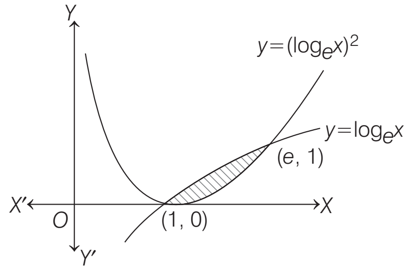
$$= 2\int_0^{1/2} \left[(x - 1)^2 - \frac{1}{4}\right] dx$$

$$= 2\left[\frac{(x - 1)^3}{3} - \frac{x}{4}\right]_0^{1/2} = 2\left[-\frac{1}{24} - \frac{1}{8} + \frac{1}{3}\right]$$

$$= \frac{1}{3} \text{ sq unit}$$

35 Required area, *A*

$$= \int_1^e [\log x - (\log x)^2]\, dx$$



$$A = \int_1^e \log x\, dx - \int_1^e (\log x)^2 dx$$

$$= [x\log x - x]_1^e - [x(\log x)^2 - 2(x\log x - x)]_1^e$$

$$= [e - e - (-1)] - [e(1)^2 - 2e + 2e - (2)]$$

$$= 1 - (e - 2) = 3 - e$$

SESSION 2

1 $h(x) = f(\cos x) = \cos^2 x$

$$18x^2 - 9\pi x + \pi^2 = 0$$

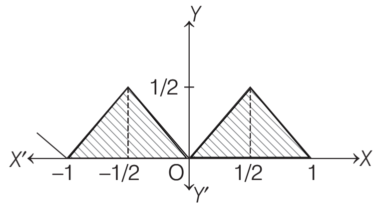
$$\Rightarrow \quad (6x - \pi)(3x - \pi) = 0$$

$$\Rightarrow \quad x_1 = \pi/6,\ x_2 = \pi/3$$

$\therefore$ Required area $= \int_{\pi/6}^{\pi/3} \cos^2 x\, dx = \pi/12$

2 Graph of

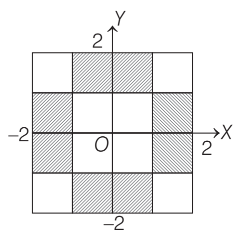
$f(x) = \min(|x|, |x - 1|, |x + 1|)$ is



Required area

$$= 2 \times \left[\frac{1}{2} \times 1 \times \frac{1}{2}\right] = \frac{1}{2} \text{ sq unit}$$

3



$[|x|] + [|y|] = 1$

$\Rightarrow \quad [|x|] = 1, [|y|] = 0$

or $\quad [|x|] = 0, [|y|] = 1$

$\Rightarrow \quad 1 \le |x| < 2, 0 \le |y| < 1$

or $0 \le |x| < 1$ or $1 \le |y| < 2$

$\Rightarrow x \in (-2, -1] \cup [1, 2), y \in (-1, 1)$

or $x \in (-1, 1), y \in (-2, -1] \cup [1, 2)$

$\therefore$ Required area $= 8 \times 1 \times 1 = 8$

4 In the equation of curve $xy^2 = 4(2 - x)$, the degree of y is even. Therefore, the curve is symmetrical about X-axis and lies in $0 < x \le 2$.

The area bounded by the curve and the Y-axis is $2\int_0^2 y\,dx$

$$= 2\int_0^2 2\sqrt{\frac{2-x}{x}}\,dx = 4\int_0^2 \sqrt{\frac{2-x}{x}}\,dx$$

Put $x = 2\sin^2\theta \Rightarrow dx = 4\sin\theta \cdot \cos\theta\, d\theta$

$$\therefore \text{Required area} = 4\int_0^{\pi/2} \sqrt{\frac{2 - 2\sin^2\theta}{2\sin^2\theta}} \cdot 4\sin\theta \cdot \cos\theta\, d\theta$$

$$= 8\int_0^{\pi/2} 2\cos^2\theta\, d\theta$$

$$= 8\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= 8\left[\theta + \sin\frac{2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 8\left[\frac{\pi}{2} + 0 - 0\right] = 4\pi$$

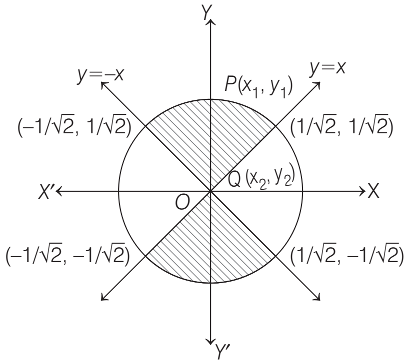
5 Required area = Area of the shaded region

$= 4$ (Area of the shaded region in first quadrant)

$$= 4\int_0^{1/\sqrt{2}} (y_1 - y_2)\,dx$$

$$= 4\int_0^{1/\sqrt{2}} (\sqrt{1 - x^2} - x)\,dx$$

$$= 4\left[\frac{1}{2}x\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1} x - \frac{x^2}{2}\right]_0^{1/\sqrt{2}}$$



$$= 4\left[\frac{1}{2\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{\pi}{4} - \frac{1}{4}\right]$$

$$= \frac{\pi}{2} \text{ sq units}$$

6 $(y - 2)^2 = x - 1$ ...(i)

Curve (i) is a parabola with vertex at the point $A(1, 2)$, axis $y - 2 = 0$ i.e. $y = 2$ and concavity towards positive X-axis.

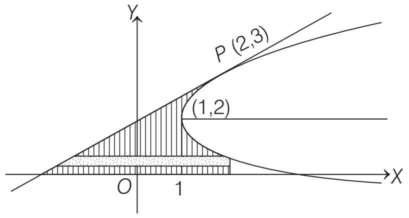
For the point, say P, at which ordinate $y = 3$ and $x = 2$

Equation of tangent at $P(2, 3)$ is

$$y - 3 = \left[\frac{dy}{dx}\right]_{(2,3)} (x - 2)$$

or $\quad y - 3 = \frac{1}{2}(x - 2)$ i.e.

$x - 2y + 4 = 0$ ...(ii)

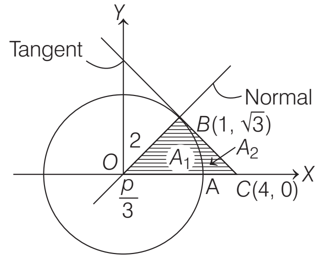


$$\therefore \text{ Area} = \int_0^3 [y^2 - 4y + 5 - (2y - 4)]\,dy$$

$$= \left[\frac{y^3}{3} - 3y^2 + 9y\right]_0^3$$

$$= 9 - 27 + 27 - 0 = 9.$$

7



Let area of portion $OAB = A_1$

and area of portion $ABC = A_2$

The equation of tangent at $(1, \sqrt{3})$ is

$x + \sqrt{3}y = 4$

[$\because xx_1 + yy_1 = a^2$ is the tangent for the circle $x^2 + y^2 = a^2$ at (x_1, y_1)]

Now, the area of the

$$\Delta OBC = \frac{1}{2} \times OB \times BC$$

$$= \frac{1}{2} \times 2 \times 2\sqrt{3} = 2\sqrt{3}$$

The area of portion OAB i.e. $A_1 = \frac{r^2\theta}{2}$

$$= \frac{4 \cdot \pi/3}{2} = \frac{2\pi}{3}.$$

Now, $A_2 = \Delta OBC - OAB = 2\sqrt{3} - 2\pi/3$

$$\frac{A_1}{A_2} = \frac{2\pi/3}{\frac{6\sqrt{3} - 2\pi}{3}}$$

$$= \frac{2\pi}{6\sqrt{3} - 2\pi} = \frac{\pi}{3\sqrt{3} - \pi}.$$

8 On solving $x^2 = 4y$ and $x + y = 3$

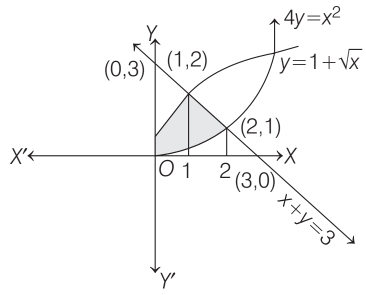
we get, $\frac{x^2}{4} + x = 3$

$\Rightarrow \quad x^2 + 4x - 12 = 0$

$\Rightarrow \quad (x + 6)(x - 2) = 0 \Rightarrow x = 2, y = 1$

Solving $y = 1 + \sqrt{x}$ and $y = 3 - x$,

we get $1 + \sqrt{x} = 3 - x \Rightarrow x = 1, y = 2$



Required Area

$$\int_0^1 (1 + \sqrt{x})\,dx + \int_1^2 (3 - x)\,dx - \int_0^2 \frac{x^2}{4}\,dx$$

$$\left[x + \frac{2}{3}x^{3/2}\right]_0^1 + \left[3x - \frac{x^2}{2}\right]_1^2 - \left[\frac{x^3}{12}\right]_0^2$$

$$= \left(1 + \frac{2}{3}\right) + \left[(6 - 2) - \left(3 - \frac{1}{2}\right)\right] - \left[\frac{8}{12}\right]$$

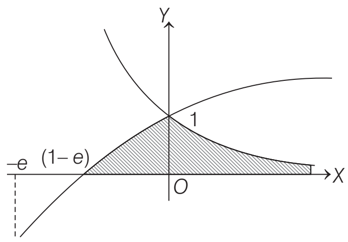
$$= \frac{5}{2}$$

9 $\quad y = \log_e (x + e)$ ...(i)

$\quad y = e^{-x}$ ...(ii)

Required area

$$= \int_{1-e}^0 \log(x + e)\,dx + \int_0^\infty e^{-x}\,dx$$



$$= [x\log(x + e)]_{1-e}^0 - \int_{1-e}^0 \frac{x}{x + e}\,dx + [-e^{-x}]_0^\infty$$

$$= 0 + 1 - [x - e\log(x + e)]_{1-e}^0$$

$$= 1 + 1 = 2$$

10 $f(x/y) = f(x) - f(y)$ ...(i)

$x = y = 1 \Rightarrow f(1) = 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

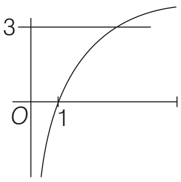
$$= \lim_{h \to 0} \frac{f\left(\frac{x + h}{x}\right)}{h} \text{ [using Eq. (i)]}$$

$$= \lim_{h \to 0} \frac{f(1 + h/x)}{h/x} \cdot \frac{1}{x} = \frac{3}{x}$$

$\Rightarrow f(x) = 3\ln x + c$

$f(1) = 0 \Rightarrow c = 0.$

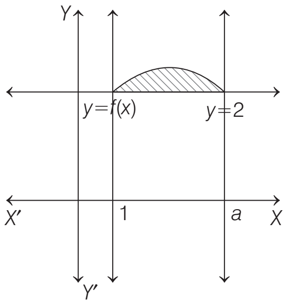
$\therefore \quad y = f(x) = 3\log_e x$



Required area $= \int_{-\infty}^{3} e^{y/3}\, dy$

$= 3[e^{y/3}]_{-\infty}^{3} = 3e$

11 According to given condition, we have



$$\int_1^a [f(x) - 2]dx = \frac{2}{3}[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}]$$

On differentiating both sides w.r.t. a, we get

$$f(a) - 2 = \frac{2}{3}\left[\frac{3}{2}(2a)^{1/2} \cdot 2 - 3\right]$$

$\Rightarrow \quad f(a) - 2 = 2\sqrt{2a} - 2$

$\Rightarrow \quad f(a) = 2\sqrt{2a}$

$\Rightarrow \quad f(x) = 2\sqrt{2x},\ x \ge 1$

12 $\frac{3\pi}{2} \le x \le 2\pi \Rightarrow -2\pi \le -x \le -\frac{3\pi}{2}$

$$\cos^{-1}(\sin x) = \cos^{-1}\cos\left(2\pi + \frac{\pi}{2} - x\right)$$

$$= 2\pi + \frac{\pi}{2} - x$$

and $\sin^{-1}(-\cos x)$

$$= \sin^{-1}\sin\left(\frac{3\pi}{2} - x\right) = \frac{3\pi}{2} - x$$

$\therefore \quad y = 2\pi + \pi/2 - x + 3\pi/2 - x$

$= 4\pi - 2x$

Required area $= \int_{3\pi/2}^{2\pi} (4\pi - 2x)dx = \frac{\pi^2}{4}$

13 $f(x) - f(x/7) = x/7$

$f(x/7) - f(x/7^2) = x/7^2$

$f(x/7^2) - f(x/7^3) = x/7^3$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$f(x/7^{n-1}) - f(x/7^n) = x/7^n.$

Adding, we get

$$f(x) - f(x/7^n) = \frac{x}{7}\left(1 + \frac{1}{7} + \ldots + \frac{1}{7^{n-1}}\right)$$

$$= \frac{x}{6}\left(1 - \frac{1}{7^n}\right)$$

Taking limit as $n \to \infty$, we get

$$f(x) - f(0) = \frac{x}{6} \Rightarrow f(x) = 1 + \frac{x}{6}$$

$[\because f(0) = 1]$

Required area $= \int_{-6}^{0} \left(1 + \frac{x}{6}\right) = 3$

14 Given curves are

$y = x - bx^2$ and $by = x^2$

Solving these, we get $x = 0, b/(1 + b^2)$

$$\therefore \quad \Delta(b) = \left|\int_0^{b/b+1} \left(\frac{x^2}{b} - x + bx^2\right)dx\right|$$

$$= \left|\left[\left(\frac{b^2+1}{b}\right)\frac{x^3}{3} - \frac{x^2}{2}\right]_0^{b/(b^2+1)}\right|$$

$$= \frac{1}{6}\frac{b^2}{(b^2+1)^2}$$

$$\Delta'(b) = \frac{1}{6} \cdot \frac{2b(b^2+1)^2 - 2b^2(b^2+1) \times 2b}{(b^2+1)^4}$$

$$= \frac{2b(1-b)(1+b)}{(b^2+1)^3}$$

$\Rightarrow \quad \Delta(b)$ is max. for $b = 1, -1$.

$\left[\because \Delta'(b) = \underset{-1 \quad 0 \quad 1}{\oplus \ \ominus \ \oplus \ \ominus}\right]$

15 In $(0, \pi/4)$, $\tan x > 0$

$$\therefore A_n = \int_0^{\pi/4} \tan^n x\, dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x(\sec^2 x - 1)\,dx$$

$$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x\, dx$$

$\Rightarrow \quad A_n = \frac{1}{n-1} - A_{n-2}$

$\Rightarrow \quad A_n + A_{n-2} = \frac{1}{n-1}$

For $n > 2$, $0 < x < \pi/4$

$\Rightarrow 0 < \tan x < 1 \Rightarrow \tan^{n-2} x > \tan^n x$

$\Rightarrow \quad A_{n-2} > A_n$

$\Rightarrow \quad 2A_n < \frac{1}{n-1}$

$\Rightarrow \quad A_n < \frac{1}{2n-2}$

Also, $A_{n+2} + A_n = \frac{1}{n+1}$

$\Rightarrow \quad 2A_n > \frac{1}{n+1}$

$\therefore \quad \frac{1}{2n+2} < A_n < \frac{1}{2n-2}$